Ockham algebras with double pseudocomplementation

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An *Ockham algebra* is a bounded distributive lattice L together with a dual endomorphism $f : L \to L$.

A *double p-algebra* (or *lattice with double pseudocomplementation*) is a lattice L with a smallest element 0 and a biggest element 1 with two mappings $^* : L \to L$ and $^+ : L \to L$ such that $x \wedge y = 0 \iff y \leq x^*$ and $x \vee y = 1 \iff y \geq x^+$.

In [**6**], Katriňák characterised a subdirectly irreducible distributive double p-algebra that is given as follows:

Theorem A ([**6**, Theorem 4). *Let L be a distributive double p-algebra and let $|L| \geq 3$. Then L is subdirectly irreducible if and only if*

(i) L is nearly regular, namely $|[a]G| \leq 2$ for every $a \in L$;

(ii) $Z(L) = \{0, 1\}$;

(iii) If L is regular then there exists $1 \neq d \in D(L) \equiv \{x \in L | x^* = 0\}$ such that $x^{n(+*)} \leq d$ for all $1 \neq x \in D(L)$ and some $n \in \mathbf{N}$;

(iv) If L is not regular then for all $1 \neq x \in D(L)$ with $|[x]G| = 1$ there exists $d \in D(L)$ with $|[d]G| \neq 1$ such that $x^{n(+*)} \leq d$ for some $n \in \mathbf{N}$.

The conditions (i)-(iv) are independent.

In this work we consider class of algebras that is contained in both the class $\mathbf{O}$ of Ockham algebras and the class of double p-algebras. This subvariety is defined as follows.

Definition By a *double pseudocomplemented Ockham algebra* $(L; \wedge, \vee, f, ^*, ^+, 0, 1)$ (shortly, double **pO**-algebra) we shall mean a bounded distributive lattice $(L;\wedge,\vee,0,1)$ together with three unary operations, denoted by f, * and $+$, such that

(1) $(L;f)\in\mathbf{O}$;

(2) $(L;^*)\in\mathbf{p}$ (the class of pseudocomplemented algebras);

(3) $(L;^+)\in\mathbf{p}^+$ (the class of dual pseudocomplemented algebras);

(4) f and * commute;

(5) f and $^+$ commute.

We shall denote by **DPO** the class of double **pO**-algebras. The following basic property is clear.

Theorem 1 Let $L\in\mathbf{DPO}$ then, the following properties hold:

(1) $(\forall a\in L)\ f(a^{**})=f(a^{++})=f(a)$;

(2) $(\forall a\in L)\ f(a^{n(+*)})=f(a^{n(*+)})=f(a)$ for all $n\geq 1$;

(3) $(\forall a\in L)\ f(a^*)=f(a^+)$.

By a *congruence* on a **DPO**-algebra $(L;f,^*,^+)$ we mean a lattice congruence ϑ such that

$$(a,b)\in\vartheta \ \Rightarrow\ (f(a),f(b))\in\vartheta,\ (a^*,b^*)\in\vartheta \text{ and } (a^+,b^+)\in\vartheta.$$

Clearly, Φ defined by

$$(x,y)\in\Phi \iff f(x)=f(y)$$

and G defined by

$$(x,y)\in G \iff x^*=y^* \quad\text{and}\quad x^+=y^+$$

are (double **pO**-algebra) congruences, and it is also clear that $G\leq\Phi$.

Given $a,b\in L$ with $a\leq b$, we denote by $\theta(a,b)$ the smallest congruence on L that identifies a and b, by $\theta_{lat}(a,b)$ the smallest lattice congruence on L that identifies a and b. We recall (see [**3**]) that in an Ockham algebra $(L;f)$ the smallest congruence that identifies a and b is

$$\theta_f(a,b)=\bigvee_{n\geq 0}\theta_{lat}(f^n(a),f^n(b))$$

and (see [**6**]) that in a double p-algebra $(L;^*,^+)$ the smallest congruence that identifies a and b is

$$\theta_{dp}(a,b)=\theta_{lat}(a,b)\vee\bigvee_{n\geq 0}[\theta_{lat}((a^*\wedge b)^{*n(+*)},1)\vee\theta_{lat}(0,(a\vee b^+)^{+n(*+)})].$$

The principal congruences on **DPO**-algebra can be described as follows:

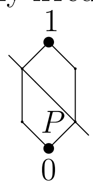
Theorem 2 Let $(L; f,^*,^+) \in \mathbf{DPO}$ and let $a, b \in L$ be such that $a \leq b$. Then

$$(\star) \qquad \theta(a,b) = \theta_f(a,b) \vee \theta_{dp}(a,b).$$

The following is our main result on a description of the structure of a subdirectly irreducible $\mathbf{DPK}_{1,1}$-algebra $(L; f,^*,^+)$:

Theorem 3 Let $L \in \mathbf{DPK_{1,1}}$ with $D(L) \neq \{1\}$ be subdirectly irreducible. Then L is of one of the following forms:

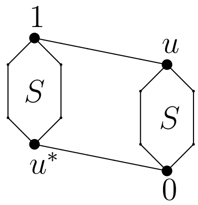
(1) If L has no coatom x such that $x \parallel D(L) \setminus \{1\}$, then as a double p-algebra $(L;^*,^+)$ is also subdirectly irreducible. L consists of two Φ-classes:



namely, P and $L \setminus P$, where P is a prime ideal of L, $L \setminus P$ is a prime filter of L.

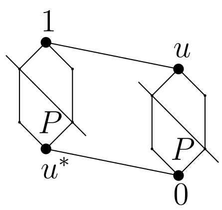
(2) If L has only one coatom u such that $u \parallel D(L) \setminus \{1\}$ then as a double p-algebra $L \simeq \mathbf{2} \times S$ where $S \equiv ([0,u];^{\bar{*}},^{\bar{+}})$ is subdirectly irreducible, and there are two possibilities:

(a) $f(u) = 1$, in this case L consists of two Φ-classes



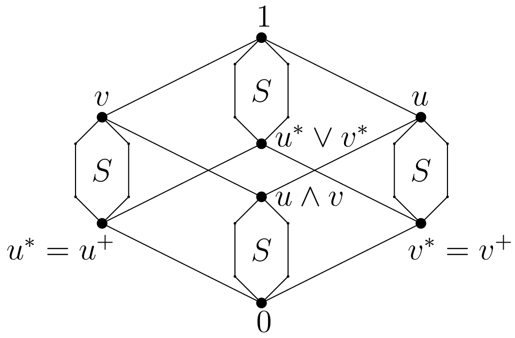
namely, $[0,u]$ and $[u^*,1]$.

(b) u is a fixed point, in this case L consists of four Φ-classes



namely, a prime ideal P of S, $Q = [0, u] \setminus P$, $u^* \vee P$ and $u^* \vee Q$.

(3) If L has two coatoms u, v such that $\{u, v\} \parallel D(L) \setminus \{1\}$, then as a double p-algebra $L \simeq \mathbf{2}^2 \times S$ where $S \equiv ([0, u \wedge v]; ^{\overline{*}}, ^{\overline{+}})$ is subdirectly irreducible. In this case, $u^* = u^+$ and $v^* = v^+$ and either u is a fixed point with $u^* = f(v)$, or v is a fixed point with $v^* = f(u)$, and L consists of four Φ-classes



namely, $[0, u \wedge v]$, $[v^*, u]$, $[u^*, v]$ and $[u^* \vee v^*, 1]$.

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